พีชคณิตของจำนวนเชิงซ้อน

ให้ $z_1 = a+bi$, $z_2 = c+di$, $\cancel{z_3 = e+fi}$

และ $k \in R$

① $z_1 = z_2$ ก็ต่อเมื่อ $a = c$ และ $b\cancel{i} = d\cancel{i}$

② $z_1 + z_2 = (a+bi) + (c+di)$

$= (a+c) + (b+d)i$

③ $z_1 - z_2 = (a+bi) - (c+di)$

$= a+bi-c-di$

$= (a-c) + (b-d)i$

④ $kz_1 = k(a+bi) = ka + kbi$

⑤ $z_1 z_2 = (a+bi)(c+di) = ac + adi + bci + bd\overset{-1}{i^2}$

$= (ac-bd) + (ad+bc)i$

⑥ $\overline{z_1} = \overline{a+bi} = a-bi$

⑦ $|z_1| = \sqrt{a^2+b^2}$

Note

$\cancel{a^2+b}$

$a^2 - b^2 = (a+b)(a-b)$

$a^2 + b^2 = (a+bi)(a-bi)$

⑧ $\frac{z_1}{z_2} = \frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di}$

$= \frac{(a+bi)(c-di)}{c^2-(di)^2} = \frac{(a+bi)(c-di)}{c^2+d^2}$

สมบัติ

1. $z_1 + z_2 = z_2 + z_1$ และ $z_1 \cdot z_2 = z_2 \cdot z_1$
2. $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ และ

   $(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$
3. $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$
4. $-(-z_1) = z_1$ และ $z^{-1} = \frac{1}{z}$ $\therefore (z^{-1})^{-1} = z$
5. $\cancel{\overline{z_1 + z_2}}$ $\overline{\overline{z_1}} = z_1$
6. $\overline{z_1 \substack{+\\-\\\times\\\div} z_2} = \overline{z_1} \substack{+\\-\\\times\\\div} \overline{z_2}$
7. $\overline{z^n} = (\overline{z})^n$
8. $|z| = |-z| = |\overline{z}| = |-\overline{z}|$
9. $|z_1 \cdot z_2| = |z_1||z_2|$
10. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ เมื่อ $z_2 \neq 0$

* 11. $z \cdot \overline{z} = |z|^2 \Rightarrow$ ~~(a+bi)~~

   $(a+bi)(a-bi) = a^2 + b^2$
12. $|z_1^n| = |z_1|^n$

* แบบฝึกหัดทบทวน

$\boxed{3}$ $z_1 = 2+3i$, $z_2 = -4+i$, $z_3 = 3-4i$, $z_4 = -3+3i$

3.1) $z_1 + z_2 = (2+3i)+(-4+i) = -2+4i$

3.2) $z_2 - z_3 = (-4+i)-(3-4i) = -4+i-3+4i = -7+5i$

3.3) $5z_3 = 5(3-4i) = 15-20i$

3.4) $z_1 z_4 = (2+3i)(-3+3i) = -6+6i-9i-9 = -15-3i$

3.5) $z_4(z_1+z_2) = (-3+3i)(-2+4i)$

$= 6-12i-6i-12 = -6-18i$

3.6) $(z_4)^2 = (-3+3i)^2 = 9-18i+9i^2 = 9-18i-9 = -18i$

3.7) $\dfrac{z_3}{z_4} = \dfrac{3-4i}{-3+3i} \cdot \dfrac{-3-3i}{-3-3i} = \dfrac{(3-4i)(-3-3i)}{(-3)^2-(3i)^2}$

$= \dfrac{-9-9i+12i-12}{9+9} = \dfrac{-21+3i}{18} = \dfrac{-21}{18} + \dfrac{3i}{18}$

$= \dfrac{-7}{6} + \dfrac{1}{6}i$

3.8) $(z_2 - 3z_1)^{-1}$

หา $z_2 - 3z_1 = -4+i-3(2+3i) = -4+i-6-9i$

$= -10-8i$

$(z_2-3z_1)^{-1} = (-10-8i)^{-1} = \dfrac{1}{-10-8i} \cdot \dfrac{-10+8i}{-10+8i}$

$= \dfrac{-10+8i}{100+64} = \dfrac{-10+8i}{164} = \dfrac{-10}{164} + \dfrac{8i}{164} = \dfrac{-5}{82} + \dfrac{2i}{41}$

3.9) $\overline{z_1+z_3} = \overline{(+2+3i)+(3-4i)} = \overline{5-i} = 5+i$

3.10) $\overline{(z_2-\overline{z_4})} = \overline{z_2} - \overline{(\overline{z_4})} = \overline{z_2} - z_4$

$= \overline{-4+i} - (-3+3i) = -4-i+3-3i$

$= -1-4i$

3.11) $(\overline{z_1})^{-1} = (\overline{2+3i})^{-1} = (2-3i)^{-1} = \frac{1}{2-3i} \cdot \frac{2+3i}{2+3i}$

~~3.12) $|z_2 z_3|$ =~~ $= \frac{2+3i}{4+9} = \frac{2+3i}{13} = \frac{2}{13} + \frac{3}{13}i$

3.12) $|z_2 z_3| = |z_2||z_3| = \sqrt{(-4)^2+1^2}\sqrt{3^2+(-4)^2}$

$= (\sqrt{17})(5) = 5\sqrt{17}$

3.13) $|(4z_1)^2| = |4^2 z_1^2| = |4^2||z_1^2| = 16|z_1|^2$

$= 16\left(\sqrt{2^2+3^2}\right)^2 = 16(13) = 208$

3.14) $\left|\frac{z_2+z_3}{z_4}\right| = \frac{|z_2+z_3|}{|z_4|} = \frac{|-1-3i|}{|-3+3i|} = \frac{\sqrt{(-1)^2+(-3)^2}}{\sqrt{(-3)^2+(+3)^2}}$

$= \frac{\sqrt{10}}{\sqrt{18}} = \sqrt{\frac{\cancel{10}\,5}{\cancel{18}\,9}} = \frac{\sqrt{5}}{3}$

3.15) $|\overline{z_4} z_3|^{-2} = (|\overline{z_4}||z_3|)^{-2} = (\sqrt{18}(5))^{-2}$

จาก $|\overline{z_4}| = \sqrt{(-3)^2+(-3)^2}$

$|z_4| = \sqrt{18}$

$= \frac{1}{(5\sqrt{18})^2} = \frac{1}{5^2(18)} = \frac{1}{450}$

$\underbrace{5\times5\times2}\times9$

$\boxed{4}$ ให้ $z = a+bi$ จงพิสูจน์ว่า $\boxed{|z|^2} \stackrel{?}{=} \boxed{z\cdot\bar{z}}$

พิสูจน์ $|z|^2 = \left(\sqrt{a^2+b^2}\right)^2 = a^2+b^2$

$$z\cdot\bar{z} = (a+bi)(a-bi) = a^2-(bi)^2 = a^2+b^2$$

$$\therefore\ |z|^2 = z\cdot\bar{z}$$

$\boxed{5}$ กำหนดให้ z_1 และ z_2 เป็นจำนวนเชิงซ้อน

จงพิสูจน์ว่า $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $z_2 \neq 0$

$|\square|^2 = |\Delta|^2$
$\therefore\ \square = \Delta$
หรือ $\square = -\Delta$

จาก $|z|^2 = z\cdot\bar{z}$

พิสูจน์ $\left|\dfrac{z_1}{z_2}\right|^2 = \dfrac{z_1}{z_2}\cdot\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{z_1}{z_2}\cdot\dfrac{\bar{z}_1}{\bar{z}_2}$

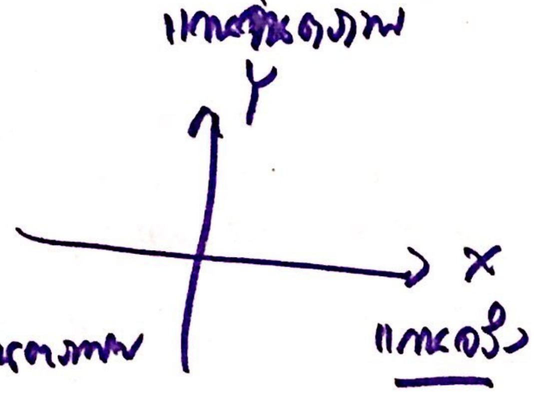
$$= \frac{z_1\cdot\bar{z}_1}{z_2\cdot\bar{z}_2} = \frac{|z_1|^2}{|z_2|^2} = \left(\frac{|z_1|}{|z_2|}\right)^2$$

$\therefore\ \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ หรือ $\left|\dfrac{z_1}{z_2}\right| = -\dfrac{|z_1|}{|z_2|}$

$\because$ ค่าสัมบูรณ์ของจำนวนเชิงซ้อนมีค่าไม่ติดลบ

ดังนั้น $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

** กราฟของจำนวนเชิงซ้อน



$z = (x,y)$

$= x + yi$

x ส่วนจริง, y ส่วนจินตภาพ

→ เชื่อมโยงกับ เรขาคณิตวิเคราะห์

แบบฝึกหัดทบทวน

$\boxed{6}$ จงเขียนกราฟของจำนวนเชิงซ้อน

6.1) $|z+1-2i| \leq 2$

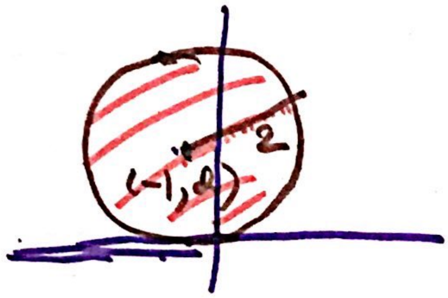
ให้ $z = x+yi = (x,y)$

$|x+yi+1-2i| \leq 2$

$|(x+1)+(y-2)i| \leq 2$

$\sqrt{(x+1)^2+(y-2)^2} \leq 2$

$(x+1)^2+(y-2)^2 \leq 2^2$



6.2) $|z-1| > |z-i|$

ให้ $z = x+yi = (x,y)$

$|x+yi-1| > |x+yi-i|$

$|(x-1)+yi| > |x+(y-1)i|$

$\sqrt{(x-1)^2+y^2} > \sqrt{x^2+(y-1)^2}$

$(x-1)^2+y^2 > x^2+(y-1)^2$

$$(x-1)^2 - x^2 > (y-1)^2 - y^2$$

$$(x-1+x)(\cancel{x}-1-\cancel{x}) > (y-1+y)(y-1-y)$$

$$-(2x-1) > -(2y-1)$$

$$2x-1 < 2y-1$$

$$x < y$$